Exponential and Logistic Models

Math 130 - Essentials of Calculus

26 March 2021

Compound Interest

If a savings account earns an annual interest rate of r (expressed as a decimal, not a percentage), then the future value of the account after t years with an initial investment of P dollars would be

$$A(t) = P(1+r)^t.$$

Compound Interest

If a savings account earns an annual interest rate of r (expressed as a decimal, not a percentage), then the future value of the account after t years with an initial investment of P dollars would be

$$A(t) = P(1+r)^t.$$

More typically, you will have a compounding period of less than a year, such as monthly or quarterly. If the compounding happens n times per year (e.g., $n = 4$ for quarterly compounding), then the interest rate per quarter will be given by $\frac{r}{n}$, where r is still given as a yearly rate. In this case, the future value after t years will be

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}.$$

Compounding Continuously

It is also possible to increase the compounding to happen at every instant of time, which would correspond to taking the limit as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} A(t) &= \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} \\
&= \lim_{n\to\infty} P\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} \\
&= P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} \\
&= P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{rt} \\
&= Pe^{rt}
\end{aligned}$$

Example

Example

If \$3000 is invested at 5% interest, find the value of the investment if interest is compounded

1. *annually*
2. *quarterly*
3. *monthly*
4. *continuously*

How long will it take for the value of the investment to double if the interest is compounded quarterly?

Exponential Growth

In the situation where a quantity changes at a constant percentage rate, we can model the situation with a *differential equation*:

$$A'(t) = k \cdot A(t)$$

which says that the rate of change in A is equal to k times A.

Exponential Growth

In the situation where a quantity changes at a constant percentage rate, we can model the situation with a *differential equation*:

$$A'(t) = k \cdot A(t)$$

which says that the rate of change in A is equal to k times A. The solution to this differential equation is given by

$$A(t) = Ce^{kt}$$

where C is the *initial value* or initial quantity. k represents the *relative growth rate*.

Exponential Growth

Example

A bacteria culture initially contains 100 *cells and grows at a rate proportional to its size. After an hour, the population has increased to* 420.

1. *Find the relative growth rate.*
2. *Find an expression for the number of bacteria after t hours.*
3. *Find the number of bacteria after three hours.*
4. *Find the rate of growth after three hours.*
5. *When will the population reach* 10,000*?*

Now You Try It!

Example

The half-life of the radioactive material cesium-137 is 30 *years. Suppose we have a* 100*mg sample.*

1. *Find the relative growth rate.*
2. *Write a formula that gives the mass that remains after t years.*
3. *How much of the sample remains after* 100 *years?*
4. *After how long will only* 1*mg remain?*
5. *At what rate is the mass decreasing after* 100 *years?*

Logistic Growth

Sometimes a population increases exponentially at first, but then levels off when approaching its maximum population, called the *carrying capacity*, that the environmental conditions can sustain.

Logistic Growth

Sometimes a population increases exponentially at first, but then levels off when approaching its maximum population, called the *carrying capacity*, that the environmental conditions can sustain. The model for a situation like this is called a *logistic function* which has the form

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

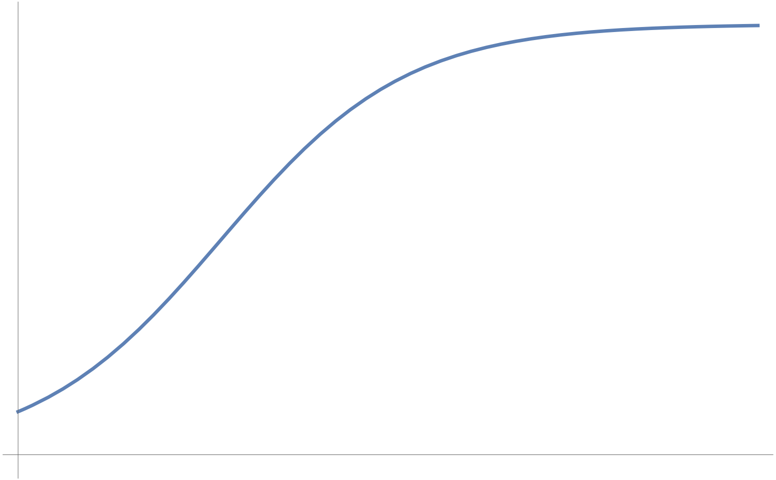
where M is the carrying capacity, t is time, k is a constant, and A is a constant given by

$$A = \frac{M - P_0}{P_0}$$

where P_0 is the *initial population*.

Logistic Growth

Here is the graph of a typical logistic function



Logistic Growth

Logistic growth also satisfies a differential equation given by

$$P'(t) = kP(t)\left(1 - \frac{P(t)}{M}\right).$$

The meaning of this equation is that the rate of change of the population is proportional to the product of the population and how far the population is from the carrying capacity.

Logistic Growth

Logistic growth also satisfies a differential equation given by

$$P'(t) = kP(t)\left(1 - \frac{P(t)}{M}\right).$$

The meaning of this equation is that the rate of change of the population is proportional to the product of the population and how far the population is from the carrying capacity. Looking at the equation, we can see that $P'(t)$ is close to zero when $P(t)$ is close to zero, or $P(t)$ is close to M.

EXAMPLE

EXAMPLE

A lake is stocked with 1000 *fish and the fish population is expected to follow the model*

$$P(t) = \frac{17,000}{1 + 16e^{-0.7t}}$$

where t is the time elapsed, in years.

1. *What is the carrying capacity?*
2. *What is the fish population after* 2.5 *years?*
3. *How many years are required for the fish population to reach* 12,000*?*
4. *What is the growth rate of the fish population after five years?*

Example

Example

The number of mountain lions in a wildlife preserve is modeled by

$$P(t) = \frac{1680}{1 + 4.2e^{-0.11t}}$$

where t is the number of years after January 1, 2010.

1. *What is the carrying capacity? How many mountain lions are there on January 1, 2010?*
2. *According to the model, what is the population after 15 years?*
3. *When does the model predict that the mountain lion population will reach 1500?*
4. *Compute and interpret $P'(12)$.*